Solutions to home work week 1

1. For an ideal gas, the internal energy $U(T)$ only depends on the temperature. Thus, for reversible expansion (or compression) from V_1 to V_2 at constant temperature T, $dU = 0$ and $dw = -pdV$. Integrating both sides of this equation:

$$\int_{w_1}^{w_2} dw = -\int_{V_1}^{V_2} pdV \tag{1}$$

Because for one mole of an ideal gas $pV = RT$, we can substitute $p = \frac{RT}{V}$ in expression 1, we obtain:

$$\begin{aligned} \textstyle\int_{w_1}^{w_2} dw &= -RT \textstyle\int_{V_1}^{V_2} \frac{1}{V} dV \\ \Delta w &= -RT \ln \frac{V_2}{V_1} \end{aligned} \tag{2}$$

If we expand and then compress again at the same temperature, and do this *reversibly*, we first let the gas perform work $\Delta w = -RT \ln \frac{V_2}{V_1}$, and then perform work on the gas $\Delta w = -RT \ln \frac{V_1}{V_2}$. Because $\ln x = -\ln \frac{1}{x}$, the work done on the gas is $RT \ln \frac{V_2}{V_1}$. Adding these two works yields a total of zero work: $\Delta w^{\text{tot}} = 0$

However, if the compression is performed at lower temperature T_{c}, we have

$$\begin{aligned} \Delta W &= -RT_{\text{h}} \ln \frac{V_2}{V_1} - RT_{\text{c}} \ln \frac{V_1}{V_2} \\ &= -R \ln \frac{V_2}{V_1} (T_{\text{h}} - T_{\text{c}}) \end{aligned} \tag{3}$$

Thus, if T_{c} is below T_{h}, negative work is performed by the gas in this cycle, meaning we extracted heat from the hotter environment, turned some of that heat into work on the environment, and released the rest of that heat into the colder environment.

If the work is done irreversibly against a constant pressure, p_0, where we can assume that after the expansion the pressure inside the cylinder is the same as the external pressure p_0, we get

$$\Delta W = -p_0(V_2 - V_1) = -\frac{RT}{V_2}(V_2 - V_1) = -RT\left(1 - \frac{V_1}{V_2}\right) \tag{4}$$

2. If an ideal gas is expanded in isolation, $dQ = 0$, because there is no exchange of heat possible with the environment. During such expansion the temperature drops from T_1 to T_2. Because the internal energy is a function of temperature, we can use the heat-capacity at constant volume to compute the change in internal energy:

$$\int dU = \Delta U = \int_{T_1}^{T_2} \frac{dU}{dT} dT = C_V \int_{T_1}^{T_2} dT = C_V\,(T_2 - T_1) \tag{5}$$

   Because $dQ = 0$ and the expansion is reversible, $\Delta U = \Delta W$.

3. For ideal gas $c_V = \frac{dU}{dT}$ and $c_p = \frac{dU}{dT} + R$. From the equation of state for one mole of ideal gas $pV = RT$, we have for a infinitesimally small change in temperature:

$$dT = \frac{1}{R}\left[pdV + Vdp\right] \tag{6}$$

   Because of adiabaticity ($dQ = 0$), we have for the change in internal energy

$$dU = -pdV = c_V dT \tag{7}$$

   isolating and equating the dT terms in both expressions gives

$$-\frac{pdV}{c_V} = \frac{1}{R}pdV + \frac{1}{R}Vdp \tag{8}$$

Multiplying by c_V and using that $c_p = \frac{dU}{dT} + R$:

$$\begin{aligned}
-pdV &= \tfrac{c_V}{R} pdV + \tfrac{c_V}{R} V dp \\
0 &= \tfrac{c_V+R}{R} pdV + \tfrac{c_V}{R} V dp \\
-\tfrac{c_V}{R} V dp &= \tfrac{c_p}{R} pdV \qquad (9) \\
V dp &= \tfrac{c_p}{c_V} pdV \\
\tfrac{dp}{p} &= -\tfrac{c_p}{c_V} \tfrac{dV}{V}
\end{aligned}$$

to get the changes in V and p in going from (p_1, V_1, T_1) to (p_2, V_2, T_2) we integrate both sides:

$$\ln \frac{p_2}{p_1} = -\frac{c_p}{c_v} \ln \frac{V_2}{V_1} = \ln(\frac{V_1}{V_2})^{\frac{c_p}{c_V}} \qquad (10)$$

so that

$$p_2 V_2^{\frac{c_p}{c_V}} = P_1 V_1^{\frac{c_p}{c_V}} \qquad (11)$$

4.